Practice Problems for final exam

1. True or False. Justify your answer.

(a). Let $G = \mathbb{C} \setminus \{z \in \mathbb{R} : z \text{ is an integer}\}$. Suppose $f \in H(G)$ such that $|f(z) \leq 1$ for all $z \in G$. Then f is a constant.

(b). Let $G = \{0 < |z-1| < 1\}$. There exists $f \in H(G)$ such that $\lim_{z\to 1} |(z-1)^k f(z)| = \infty$, for all integer $k \geq 1$.

2. Suppose f has an essential singularity at 0, and g has an essential singularity at 0. Prove that at least one of the functions $f + g$ and fg has an essential singularity at $z = 0$.

3. Suppose f is a nonconstant entire function. Which of the following must be countably infinite?
(a). $f(\mathbb{Z})$ (b). $f(\mathbb{Q})$ (c). $f^{-1}(\mathbb{Q})$

4. Let f be a holomorphic function in the unit disk $\mathbb{D}$ that is injective and satisfies $f(0) = 0$ Prove that there exists a holomorphic function g in $\mathbb{D}$ such that $(g(z))^2 = f(z^2)$ for all $z \in \mathbb{D}$.

5. State Riemann Mapping Theorem, Runge's Theorem, Weierstrass Factorization Theorem, Mittag-Leffler's Theorem, Schwarz Reflection Principle.

6. Let $\{p_n\}_{n=1}^{\infty} \subset \mathbb{Z}^+$ be the sequence of prime numbers. Prove there exists $f \in H(\mathbb{C})$ such that $f(p_n) = p_{n+1}$ for each n.

7. Let f be a continuous function on $\{0 < |z| \leq 1\}$ that is analytic function on $\{0 < |z| < 1\}$. Assume $f(z) = 0$ for every $z = e^{i\theta}$ with $\frac{\pi}{4} < \theta < \frac{\pi}{3}$. Prove $f \equiv 0$.

8. Let $A_1 = \{z \in \mathbb{C} : 0 < |z| < 1\}$ and $A_2 = \{z \in \mathbb{C} : 1 < |z| < 2\}$. Prove A_1 and A_2 are not conformally equivalent.

9. Describe all analytic functions on $\mathbb{C} \setminus \{0\}$ with the property that

$$|f(z)| \leq C(|z|^2 + \frac{1}{|z|^{\frac{1}{2}}}) \text{ for some constant } C > 0.$$

10. Homework problems.